

Nonabelian Phenomena on D-branes

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Abstract. A remarkable feature of D-branes is the appearance of a nonabelian gauge theory in the description of several (nearly) coincident branes. This nonabelian structure plays an important role in realizing various geometric effects with D-branes. In particular, the branes' transverse displacements are described by matrix-valued scalar fields and so noncommutative geometry naturally appears in this framework. I review the action governing this nonabelian theory, as well as various related physical phenomena such as the dielectric effect, giant gravitons and fuzzy funnels.

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1. Introduction

Dirichlet branes have played a central role in all of the major advances in string theory in the past seven years [1]. One of the most interesting aspects of the physics of D-branes is the appearance of a nonabelian gauge symmetry when several D-branes are brought together. Of course, we understand that this symmetry emerges through the appearance of new massless states corresponding to open strings stretching between the D-branes [2]. Thus while the number of light degrees of freedom is proportional to N for N widely separated D-branes, this number grows like N^2 for N coincident D-branes. The nonabelian symmetry and the rapid growth in massless states are crucial elements in the statistical mechanical entropy counting for black holes [3], in Maldacena's conjectured duality between type IIB superstrings in $\text{AdS}_5 \times S^5$ and four-dimensional $\mathcal{N}=4$ super-Yang-Mills theory [4], in the development of M(atr ix)-theory as a nonperturbative description of M-theory [5].

Referring to the worldvolume theory for N (nearly) coincident D-branes as a nonabelian $U(N)$ gauge theory emphasizes the massless vector states, which one might regard as internal excitations of the branes. Much of the present discussion will focus on the scalar fields describing the transverse displacements of the branes. For coincident branes, these coordinate fields become matrix-valued appearing in the adjoint representation of the $U(N)$ gauge group. These matrix-valued coordinates then provide a natural framework where one might consider noncommutative geometry. What has become evident after a detailed study of the world-volume action governing the dynamics of the nonabelian $U(N)$ theory [6, 7] is that noncommutative geometries appear dynamically in many physical situations in string theory. One finds that D-branes of one dimension metamorphose into branes of a higher dimension through noncommutative configurations. These configurations provide hints of dualities relating gauge theories in different dimensions, and point to a symbiotic relationship between D-branes of all dimensions. The emerging picture is reminiscent of the 'brane democracy' speculations in very early investigations of D-branes [8].

One important example of this brane transmogrification is the 'dielectric effect' in which a set of D-branes are polarized into a higher dimensional noncommutative geometry by nontrivial background fields [6]. String theory seems to employ this brane expansion mechanism in a variety of circumstances to regulate spacetime singularities [9, 10]. So not only do string theory and D-branes provide a natural physical framework for noncommutative geometry, but it also seems that they may provide a surprising realization of the old speculation that noncommutative geometry should play a role in resolving the chaotic short-distance structure of spacetime quantum gravity.

An outline of this paper is as follows: First, section 2 presents a preliminary discussion of the world-volume actions governing the low energy physics of D-branes. Then section 3 discusses the extension to the nonabelian action, relevant for a system of

several (nearly) coincident D-branes. Section 4 provides an outline of the dielectric effect for D-branes. The next section describes an important application of the dielectric effect, namely, giant gravitons. These are spherical D3-branes whose motion causes them to expand in an $\text{AdS}_5 \times \text{S}^5$ background. Section 6 gives a discussion of how noncommutative geometries can arise in the description of intersecting branes. In Appendix A, we give a short discussion of various aspects of noncommutative geometry and fuzzy spheres which are relevant for the main text. Sections 2 and 4 are essentially a summary of the material appearing in ref. [6]. Section 5 describes the material in ref. [11] and section 6 describes that in refs. [12] and [13]. We direct the interested reader to these papers for more detailed presentations of the associated material.

2. World-volume D-brane actions

Within the framework of perturbative string theory, a Dp -brane is a $(p+1)$ -dimensional extended surface in spacetime which supports the endpoints of open strings [1]. The massless modes of this open string theory form a supersymmetric $U(1)$ gauge theory with a vector A_a , $9-p$ real scalars Φ^i and their superpartner fermions — for the most part, the latter are ignored throughout the following discussion. At leading order, the low-energy action corresponds to the dimensional reduction of that for ten-dimensional $U(1)$ super-Yang-Mills theory. However, as usual in string theory, there are higher order $\alpha' = \ell_s^2$ corrections — ℓ_s is the string length scale. For constant field strengths, these stringy corrections can be resummed to all orders, and the resulting action takes the Born-Infeld form [14]

$$S_{BI} = -T_p \int d^{p+1}\sigma \left(e^{-\phi} \sqrt{-\det(P[G+B]_{ab} + \lambda F_{ab})} \right) \quad (1)$$

where T_p is the Dp -brane tension and λ denotes the inverse of the (fundamental) string tension, *i.e.*, $\lambda = 2\pi\ell_s^2$. This Born-Infeld action describes the couplings of the Dp -brane to the massless Neveu-Schwarz fields of the bulk closed string theory, *i.e.*, the (string-frame) metric $G_{\mu\nu}$, dilaton ϕ and Kalb-Ramond two-form $B_{\mu\nu}$. The interactions with the massless Ramond-Ramond (RR) fields are incorporated in a second part of the action, the Wess-Zumino term[15, 16, 17]

$$S_{WZ} = \mu_p \int P \left[\sum C^{(n)} e^B \right] e^{\lambda F}. \quad (2)$$

Here $C^{(n)}$ denote the n -form RR potentials. Eq. (2) shows that a Dp -brane is naturally charged under the $(p+1)$ -form RR potential with charge μ_p , and supersymmetry dictates that $\mu_p = \pm T_p$. If we consider the special case of the D0-brane (a point particle), the Born-Infeld action reduces to the familiar world-line action of a point particle, where the action is proportional to the proper length of the particle trajectory. Actually this string

theoretic D0-brane action is not quite this simple geometric action, rather it is slightly embellished with the additional coupling to the dilaton which appears as a prefactor to the standard Lagrangian density. (Note, however, that the tensors B and F drop out of the action since the determinant is implicitly over a one-dimensional matrix.) Turning to the Wess-Zumino action, we see that a D0-brane couples to $C^{(1)}$ (a vector). Then eq. (2) reduces to the familiar coupling of a Maxwell field to the world-line of a point particle, *i.e.*,

$$\mu_0 \int P [C^{(1)}] \simeq q \int A_\mu \frac{dx^\mu}{d\tau} d\tau . \quad (3)$$

Higher dimensional D p -branes can also support a flux of $B + F$, which complicates the world-volume actions above. From eq. (2), we see that such a flux allows a D p -brane to act as a charge source for RR potentials with a lower form degree than $p+1$ [15]. Such configurations represent bound states of D-branes of different dimensions[2]. To illustrate this point, let us assume that B vanishes and then we may expand the Wess-Zumino action (2) for a D4-brane as:

$$\mu_4 \int \left(C^{(5)} + \lambda C^{(3)} \wedge F + \frac{\lambda^2}{2} C^{(1)} \wedge F \wedge F \right) . \quad (4)$$

Hence, the D4-brane is naturally a source for the five-form potential $C^{(5)}$. However, by introducing a world-volume gauge field with a nontrivial first Chern class, *i.e.*, exciting a nontrivial magnetic flux on the world-volume, the D4-brane also sources $C^{(3)}$, which is the potential associated with D2-branes. Hence a D4-brane with magnetic flux is naturally interpreted as a D4-D2 bound state. Similarly a D4-brane, which supports a gauge field with a nontrivial second Chern class, will source the vector potential $C^{(1)}$ and is interpreted as a D4-D0 bound state.

In both of the pieces of the D-brane action, the symbol $P[\dots]$ denotes the pull-back of the bulk spacetime tensors to the D-brane world-volume. Hence as already alluded to above, the Born-Infeld action (1) has a geometric interpretation, *i.e.*, it is essentially the proper volume swept out by the D p -brane, which is indicative of the fact that D-branes are actually dynamical objects. This dynamics becomes more evident with an explanation of the static gauge choice implicit in constructing the above action. To begin, we employ spacetime diffeomorphisms to position the world-volume on a fiducial surface defined as $x^i = 0$ with $i = p+1, \dots, 9$. With world-volume diffeomorphisms, we then match the world-volume coordinates with the remaining spacetime coordinates on this surface, $\sigma^a = x^a$ with $a = 0, 1, \dots, p$. Now the world-volume scalars Φ^i play the role of describing the transverse displacements of the D-brane, through the identification

$$x^i(\sigma) = 2\pi\ell_s^2 \Phi^i(\sigma) \quad \text{with } i = p+1, \dots, 9. \quad (5)$$

With this identification the general formula for the pull-back reduces to

$$\begin{aligned} P[E]_{ab} &= E_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \\ &= E_{ab} + \lambda E_{ai} \partial_b \Phi^i + \lambda E_{ib} \partial_a \Phi^i + \lambda^2 E_{ij} \partial_a \Phi^i \partial_b \Phi^j . \end{aligned} \quad (6)$$

In this way, the expected kinetic terms for the scalars emerge to leading order in an expansion of the Born-Infeld action (1). Note that our conventions are such that both the gauge fields and world-volume scalars have the dimensions of $length^{-1}$ — hence the appearance of the string scale in eq. (5).

Although it was mentioned above, we want to stress that these world-volume actions are low energy effective actions for the massless states of the open and closed strings, which incorporate interactions from all disk amplitudes (all orders of tree level for the open strings). The Born-Infeld action was originally derived [14] using standard beta function techniques applied to world-sheets with a boundary [18]. In principle, they could also be derived from a study of open and closed string scattering amplitudes and it has been verified that this approach yields the same interactions to leading order [19, 20, 21]. As a low energy effective action then, eqs. (1) and (2) include an infinite number of stringy corrections, which essentially arise through integrating out the massive modes of the string — see the discussion in [22]. For example, consider the Born-Infeld action evaluated for a flat Dp-brane in empty Minkowski space

$$\begin{aligned} S_{BI} &\simeq -T_p \int d^{p+1}x \sqrt{-\det(\eta_{ab} + 2\pi\ell_s^2 F_{ab})} \\ &\simeq -T_p \int d^{p+1}x \left(1 + \frac{(2\pi\ell_s^2)^2}{4} F^2 + (2\pi\ell_s^2)^4 F^4 + (2\pi\ell_s^2)^6 F^6 + \dots \right) \end{aligned} \quad (7)$$

where the precise structure of the F^4 and F^6 terms may be found in [23] and [24], respectively. Hence as well as the standard kinetic term for the world-volume gauge field, eq. (7) includes an infinite series of higher dimension interactions, which are suppressed as long as the typical components $\ell_s^2 F_{ab}$ (referred to an orthonormal frame) are small. The square-root expression in the first line resums this infinite series and so one may consider arbitrary values of $\ell_s^2 F_{ab}$ in working with this action. However, the full effective action would also include stringy correction terms involving derivatives of the field strength, *e.g.*, $\partial_a F_{bc}$ or $\partial_a \partial_b F_{cd}$ — see, for example, [24] or [25]. None of these have been incorporated in the Born-Infeld action and so one must still demand that the variations in the field strength are relatively small, *e.g.*, components of $\ell_s \partial_a F_{bc}$ are much smaller than those of F_{ab} . Of course, this discussion extends in the obvious way to derivatives of the scalar fields Φ^i .

At this point, we should also note that the bulk supergravity fields appearing in eqs. (1) and (2) are in general functions of all of the spacetime coordinates, and so they are implicitly functionals of the world-volume scalars. In static gauge, the bulk fields are evaluated in terms of a Taylor series expansion around the fiducial surface $x^i = 0$. For example, the metric functional appearing in the D-brane action would be given by

$$G_{\mu\nu} = \exp [\lambda \Phi^i \partial_{x^i}] G_{\mu\nu}^0(\sigma^a, x^i)|_{x^i=0} \quad (8)$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Phi^{i_1} \dots \Phi^{i_n} (\partial_{x^{i_1}} \dots \partial_{x^{i_n}}) G_{\mu\nu}^0(\sigma^a, x^i)|_{x^i=0} .$$

Hence the world-volume action implicitly incorporates an infinite class of higher dimension interactions involving derivatives of the bulk fields as well. However, beyond this class of interactions incorporated in eqs. (1) and (2), once again the full effective action includes other higher derivative bulk field corrections [17, 26, 27, 28]. It is probably fair to say that the precise domain of validity of the D-brane action from the point of view of the bulk fields is poorly understood.

3. Nonabelian D-brane action

As N parallel D-branes approach each other, the ground state modes of strings stretching between the different D-branes become massless. These extra massless states carry the appropriate charges to fill out representations under a $U(N)$ symmetry. Hence the $U(1)^N$ of the individual D-branes is enhanced to the nonabelian group $U(N)$ for the coincident D-branes [2]. The vector A_a becomes a nonabelian gauge field

$$A_a = A_a^{(n)} T_n, \quad F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \quad (9)$$

where T_n are N^2 hermitian generators with $\text{Tr}(T_n T_m) = N \delta_{nm}$. Central to the following is that the scalars Φ^i are also matrix-valued transforming in the adjoint of $U(N)$. The covariant derivative of the scalar fields is given by

$$D_a \Phi^i = \partial_a \Phi^i + i[A_a, \Phi^i] . \quad (10)$$

Understanding how to accommodate this $U(N)$ gauge symmetry in the world-volume action is an interesting puzzle. For example, the geometric meaning (or even the validity) of eqs. (5) or (8) seems uncertain when the scalars on the right hand side are matrix-valued. In fact, the identification of the scalars as transverse displacements of the branes does remain roughly correct. Some intuition comes from the case where the scalars are commuting matrices and the gauge symmetry can be used to simultaneously diagonalize all of them. In this case, one interpretes the N eigenvalues of the diagonal Φ^i as representing the displacements of the N constituent D-branes — see, *e.g.*, [5]. Further the gauge symmetry may be used to simultaneously interchange any pair of eigenvalues in each of the scalars and so ensures that the branes are indistinguishable. Of course, to describe noncommutative geometries, we will be more interested in the case where the scalars do not commute and so cannot be simultaneously diagonalized.

Refs. [6] and [7] made progress in constructing the world-volume action describing the dynamics of nonabelian D-branes. The essential strategy in both of these papers was to construct an action which was consistent with the familiar string theory symmetry of

T-duality [29]. Acting on D-branes, T-duality acts to change the dimension of the world-volume [1]. The two possibilities are: (i) if a coordinate transverse to the D p -brane, *e.g.*, $y = x^{p+1}$, is T-dualized, it becomes a D($p + 1$)-brane where y is now the extra world-volume direction; and (ii) if a world-volume coordinate on the D p -brane, *e.g.*, $y = x^p$, is T-dualized, it becomes a D($p - 1$)-brane where y is now an extra transverse direction. Under these transformations, the role of the corresponding world-volume fields change as

$$(i) \Phi^{p+1} \rightarrow A_{p+1} , \quad (ii) A_p \rightarrow \Phi^p , \quad (11)$$

while the remaining components of A and scalars Φ are left unchanged. Hence in constructing the nonabelian action, one can begin with the D9-brane theory, which contains no scalars since the world-volume fills the entire spacetime. In this case, the nonabelian extension of eqs. (1) and (2) is given by simply introducing an overall trace over gauge indices of the nonabelian field strengths appearing in the action [30]. Then applying T-duality transformations on $9 - p$ directions yields the nonabelian action for a D p -brane. Of course, in this construction, one also T-dualizes the background supergravity fields according to the known transformation rules [29, 31, 32]. As in the abelian theory, the result for nonabelian action has two distinct pieces [6, 7]: the Born-Infeld term

$$S_{BI} = -T_p \int d^{p+1} \sigma \text{STr} \left(e^{-\phi} \sqrt{\det(Q^i_j)} \right. \\ \left. \times \sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + \lambda F_{ab})} \right) , \quad (12)$$

with $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ and $Q^i_j \equiv \delta^i_j + i\lambda[\Phi^i, \Phi^k]E_{kj}$; and the Wess-Zumino term

$$S_{WZ} = \mu_p \int \text{STr} \left(P \left[e^{i\lambda \mathbf{i}_\Phi \mathbf{i}_\Phi} \left(\sum C^{(n)} e^B \right) \right] e^{\lambda F} \right) . \quad (13)$$

Let us enumerate the nonabelian features of this action:

1. *Nonabelian field strength:* The F_{ab} appearing explicitly in both terms is now nonabelian, of course.
2. *Nonabelian Taylor expansion:* The bulk supergravity fields are again functions of all of the spacetime coordinates, and so they are implicitly functionals of the nonabelian scalars. In the action (12,13), these bulk fields are again interpreted in terms of a Taylor expansion as in eq. (8), however the transverse displacements are now matrix-valued. the D-brane action would be given by a nonabelian Taylor expansion
3. *Nonabelian Pullback:* As was noted in refs. [33, 34], the pullback of various spacetime tensors to the world-volume must now involve covariant derivatives of the nonabelian

scalars in order to be consistent with the $U(N)$ gauge symmetry. Hence eq. (6) is replaced by

$$P[E]_{ab} = E_{ab} + \lambda E_{ai} D_b \Phi^i + \lambda E_{ib} D_a \Phi^i + \lambda^2 E_{ij} D_a \Phi^i D_b \Phi^j . \quad (14)$$

4. *Nonabelian Interior Product*: In the Wess-Zumino term (13), i_Φ denotes the interior product with Φ^i regarded as a vector in the transverse space, *e.g.*, acting on an n -form $C^{(n)} = \frac{1}{n!} C_{\mu_1 \dots \mu_n}^{(n)} dx^{\mu_1} \dots dx^{\mu_n}$, we have

$$i_\Phi i_\Phi C^{(n)} = \frac{1}{2(n-2)!} [\Phi^i, \Phi^j] C_{ji\mu_3 \dots \mu_n}^{(n)} dx^{\mu_3} \dots dx^{\mu_n} . \quad (15)$$

Note that acting on forms, the interior product is an anticommuting operator and hence for an ordinary vector (*i.e.*, a vector v^i with values in \mathbb{R}^{9-p}): $i_v i_v C^{(n)} = 0$. It is only because the scalars Φ are matrix-valued that eq. (15) yields a nontrivial result.

5. *Nonabelian Gauge Trace*: As is evident above, both parts of the action are highly nonlinear functionals of the nonabelian fields, and so eqs. (12) and (13) would be incomplete without a precise definition for the ordering of these fields under the gauge trace. Above, $S\text{Tr}$ denotes the maximally symmetric trace [35]. To be precise, the trace includes a symmetric average over all orderings of F_{ab} , $D_a \Phi^i$, $[\Phi^i, \Phi^j]$ and the individual Φ^k appearing in the nonabelian Taylor expansions of the background fields. This choice matches that inferred from Matrix theory [36], and a similar symmetrization arises in the leading order analysis of the boundary β functions [34]. However, we should note that with this definition an expansion of the Born-Infeld term (1) does agree with the string theory to fourth order in F [35, 37], but it does not seem to capture the full physics of the nonabelian fields in the infrared limit at higher orders [38] — we expand on this point below.

Some general comments on the nonabelian action are as follows: In the Born-Infeld term (12), there are now two determinant factors as compared to one in the abelian action (1). The second determinant in eq. (12) is a slightly modified version of that in eq. (1). One might think of this as the kinetic factor, since to leading order in the low energy expansion, it yields the familiar kinetic terms for the gauge field and scalars. In the same way, one can think of the new first factor as the potential factor, since to leading order in the low energy expansion, it reproduces the nonabelian scalar potential expected for the super-Yang-Mills theory — see eq. (19) below. Further note that the first factor reduces to simply one when the scalar fields are commuting, even for general background fields.

The form of the action and in particular the functional dependence of the bulk fields on the adjoint scalars can be verified in a number of independent ways. Douglas [30] observed on general grounds whatever their form, the nonabelian world-volume actions should contain a single gauge trace, as do both eqs. (12) and (13). This

observation stems from the fact that the action should encode only the low energy interactions derivable from disk amplitudes in superstring theory. Since the disk has a single boundary, the single gauge trace arises from the standard open string prescription of tracing over Chan-Paton factors on each world-sheet boundary. Further then one may note [30] that the only difference in the superstring amplitudes between the $U(1)$ and the $U(N)$ theories is that the amplitudes in the latter case are multiplied by an additional trace of Chan-Paton factors. Hence up to commutator ‘corrections’, the low energy interactions should be the same in both cases. Hence since the background fields are functionals of the neutral $U(1)$ scalars in the abelian theory, they must be precisely the same functionals of the adjoint scalars in the nonabelian theory, up to commutator corrections. The interactions involving the nonabelian inner product in the Wess-Zumino action (13) provide one class of commutator corrections. The functional dependence on the adjoint scalars also agrees with the linearized couplings for the bulk fields derived from Matrix theory [36]. As an aside, we would add that the technology developed in Matrix theory remains useful in gaining intuition and manipulating these nonabelian functionals [39, 40]. Finally we add that by the direct examination of string scattering amplitudes using the methods of refs. [19] and [21], one can verify at low orders the form of the nonabelian interactions in eqs. (12) and (13), including the appearance of the new commutator interactions in the nonabelian Wess-Zumino action [41].

As noted above, the symmetric trace prescription is known *not* to agree with the full effective string action [38]. Rather at sixth order and higher in the world-volume field strength, additional terms involving commutators of field strengths must be added to the action [42]. The source of this shortcoming is clear. Recall from the discussion following eq. (7) that in the abelian action we have disregarded all interactions involving derivatives of the field strength. However, this prescription is ambiguous in the nonabelian theory since

$$[D_a, D_b]F_{cd} = i[F_{ab}, F_{cd}] . \quad (16)$$

It is clear that with the symmetric trace we have eliminated all derivative terms, including those antisymmetric combinations that might contribute commutators of field strengths. One might choose to improve the action by reinstating these commutators. This problem has been extensively studied and the commutator corrections at order F^6 are known [42]. Considering the supersymmetric extension of the nonabelian action [43, 24] has lead to a powerful iterative technique relying on stable holomorphic bundles [44] which seems to provide a constructive approach to determine the entire effective open string action, including all higher derivative terms and fermion contributions as well. A similar iterative procedure seems to emerge from study the Seiberg-Witten map [45] in the context of a noncommutative world-volume theory [46].

As described below eq. (2), an individual Dp -brane couples not only to the RR potential with form degree $n = p + 1$, but also to the RR potentials with

$n = p - 1, p - 3, \dots$ through the exponentials of B and F appearing in the Wess-Zumino action (2). Above in eq. (13), $i_\Phi i_\Phi$ is an operator of form degree -2 , and so world-volume interactions appear in the nonabelian action (13) involving the higher RR forms. Hence in the nonabelian theory, a Dp -brane can also couple to the RR potentials with $n = p + 3, p + 5, \dots$ through the additional commutator interactions. To make these couplings more explicit, consider the D0-brane action (for which F vanishes):

$$\begin{aligned}
S_{CS} = \mu_0 \int \text{STr} \left(P \left[C^{(1)} + i\lambda i_\Phi i_\Phi (C^{(3)} + C^{(1)}B) \right. \right. \\
\left. \left. - \frac{\lambda^2}{2} (i_\Phi i_\Phi)^2 \left(C^{(5)} + C^{(3)}B + \frac{1}{2} C^{(1)}B^2 \right) \right. \right. \\
\left. \left. - i\frac{\lambda^3}{6} (i_\Phi i_\Phi)^3 \left(C^{(7)} + C^{(5)}B + \frac{1}{2} C^{(3)}B^2 + \frac{1}{6} C^{(1)}B^3 \right) \right. \right. \\
\left. \left. + \frac{\lambda^4}{24} (i_\Phi i_\Phi)^4 \left(C^{(9)} + C^{(7)}B + \frac{1}{2} C^{(5)}B^2 + \frac{1}{6} C^{(3)}B^3 + \frac{1}{24} C^{(1)}B^4 \right) \right] \right). \quad (17)
\end{aligned}$$

Of course, these interactions are reminiscent of those appearing in Matrix theory [47, 48]. For example, eq. (17) includes a linear coupling to $C^{(3)}$, which is the potential corresponding to D2-brane charge,

$$\begin{aligned}
i\lambda\mu_0 \int \text{Tr} P [i_\Phi i_\Phi C^{(3)}] \\
= i\frac{\lambda}{2}\mu_0 \int dt \text{Tr} \left(C_{tjk}^{(3)}(\Phi, t) [\Phi^k, \Phi^j] + \lambda C_{ijk}^{(3)}(\Phi, t) D_t \Phi^k [\Phi^k, \Phi^j] \right) \quad (18)
\end{aligned}$$

where we assume that $\sigma^0 = t$ in static gauge. Note that the first term on the right hand side has the form of a source for D2-brane charge. This is essentially the interaction central to the construction of D2-branes in Matrix theory with the large N limit [47, 48]. Here, however, with finite N , this term would vanish upon taking the trace if $C_{tjk}^{(3)}$ was simply a function of the world-volume coordinate t (since $[\Phi^k, \Phi^j] \in \text{SU}(N)$). However, in general these three-form components are functionals of Φ^i . Hence, while there would be no ‘monopole’ coupling to D2-brane charge, nontrivial expectation values of the scalars can give rise to couplings to an infinite series of higher ‘multipole’ moments [39].

4. Dielectric Branes

In this section, we wish to consider certain physical effects arising from the new nonabelian interactions in the world-volume action, given by eqs. (12) and (13). To begin, consider the scalar potential for Dp -branes in flat space, *i.e.*, $G_{\mu\nu} = \eta_{\mu\nu}$ with all other fields vanishing. In this case, the entire scalar potential originates in the Born-Infeld term (12) as

$$V = T_p \text{Tr} \sqrt{\det(Q^i_j)} = NT_p - \frac{T_p \lambda^2}{4} \text{Tr}([\Phi^i, \Phi^j][\Phi^i, \Phi^j]) + \dots \quad (19)$$

The commutator-squared term corresponds to the potential for ten-dimensional $U(N)$ super-Yang-Mills theory reduced to $p + 1$ dimensions. A nontrivial set of extrema of this potential is given by taking the $9 - p$ scalars as constant commuting matrices, *i.e.*,

$$[\Phi^i, \Phi^j] = 0 \quad (20)$$

for all i and j . Since they are commuting, the Φ^i may be simultaneously diagonalized and as discussed above, the eigenvalues are interpreted as the separated positions of N fundamental Dp -branes in the transverse space. This solution reflects the fact that a system of N parallel Dp -branes is supersymmetric, and so they can sit in static equilibrium with arbitrary separations in the transverse space [1].

From the results described in the previous section, it is clear that in going from flat space to general background fields, the scalar potential is modified by new interactions and so one should reconsider the analysis of the extrema. It turns out that this yields an interesting physical effect that is a precise analog for D-branes of the dielectric effect in ordinary electromagnetism. That is when Dp -branes are placed in a nontrivial background field for which the Dp -branes would normally be regarded as neutral, *e.g.*, nontrivial $F^{(n)}$ with $n > p + 2$, new terms will be induced in the scalar potential, and generically one should expect that there will be new extrema beyond those found in flat space, *i.e.*, eq. (20). In particular, there can be nontrivial extrema with noncommuting expectation values of the Φ^i , *e.g.*, with $\text{Tr}\Phi^i = 0$ but $\text{Tr}(\Phi^i)^2 \neq 0$. This would correspond to the external fields ‘polarizing’ the Dp -branes to expand into a (higher dimensional) noncommutative world-volume geometry. This is the analog of the familiar electromagnetic process where an external field may induce a separation of charges in neutral materials. In this case, the polarized material will then carry an electric dipole (and possibly higher multipoles). The latter is also seen in the D-brane analog. When the world-volume theory is at a noncommutative extremum, the gauge traces of products of scalars will be nonvanishing in various interactions involving the supergravity fields. Hence at such an extremum, the Dp -branes act as sources for the latter bulk fields.

To make these ideas explicit, we will now illustrate the process with a simple example. We consider N D0-branes in a constant background RR field $F^{(4)}$, *i.e.*, the field strength associated with D2-brane charge. We find that the D0-branes expand into a noncommutative two-sphere which represents a spherical bound state of a D2-brane and N D0-branes.

Consider a background where only RR four-form field strength is nonvanishing with

$$F_{tijk}^{(4)} = -2f\varepsilon_{ijk} \quad \text{for } i, j, k \in \{1, 2, 3\} \quad (21)$$

with f a constant (of dimensions $length^{-1}$). Since $F^{(4)} = dC^{(3)}$, we must consider the coupling of the D0-branes to the RR three-form potential, which is given above in eq. (18). If one explicitly introduces the nonabelian Taylor expansion (8), one finds the

leading order interaction may be written as

$$\frac{i}{3}\lambda^2\mu_0 \int dt \text{Tr} (\Phi^i \Phi^j \Phi^k) F_{tijk}^{(4)}(t) . \quad (22)$$

This final form might have been anticipated since one should expect that the world-volume potential can only depend on gauge invariant expressions of the background field. Given that we are considering a constant background $F^{(4)}$, the higher order terms implicit in eq. (18) will vanish as they can only involve spacetime derivatives of the four-form field strength. Combining eq. (22) with the leading order Born-Infeld potential (19) yields the scalar potential of interest for the present problem

$$V(\Phi) = NT_0 - \frac{\lambda^2 T_0}{4} \text{Tr}([\Phi^i, \Phi^j]^2) - \frac{i}{3}\lambda^2\mu_0 \text{Tr} (\Phi^i \Phi^j \Phi^k) F_{tijk}^{(4)}(t) . \quad (23)$$

Substituting in the (static) background field (21) and $\mu_0 = T_0$, $\delta V(\Phi)/\delta\Phi^i = 0$ yields

$$0 = [[\Phi^i, \Phi^j], \Phi^j] + i f \varepsilon_{ijk} [\Phi^j, \Phi^k] . \quad (24)$$

Note that commuting matrices (20) describing separated D0-branes still solve this equation. The value of the potential for these solutions is simply $V_0 = NT_0$, the mass of N D0-branes. Another interesting solution of eq. (24) is

$$\Phi^i = \frac{f}{2} \alpha^i \quad (25)$$

where α^i are any $N \times N$ matrix representation of the $SU(2)$ algebra

$$[\alpha^i, \alpha^j] = 2i \varepsilon_{ijk} \alpha^k . \quad (26)$$

For the moment, let us focus on the irreducible representation for which one finds

$$\text{Tr}[(\alpha^i)^2] = \frac{N}{3}(N^2 - 1) \quad \text{for } i = 1, 2, 3. \quad (27)$$

Now evaluating the value of the potential (23) for this new solution yields

$$V_N = NT_0 - \frac{T_0 \lambda^2 f^2}{6} \sum_{i=1}^3 \text{Tr}[(\Phi^i)^2] = NT_0 - \frac{\pi^2 \ell_s^3 f^4}{6g} N^3 \left(1 - \frac{1}{N^2}\right) \quad (28)$$

using $T_0 = 1/(g\ell_s)$. Hence the noncommutative solution (25) has lower energy than a solution of commuting matrices, and so the latter configuration of separated D0-branes is unstable towards condensing out into this noncommutative solution. One can also consider reducible representations of the $SU(2)$ algebra (26), however, one finds that

the corresponding energy is always larger than that in eq. (28). Hence it seems that the irreducible representation describes the ground state of the system.

Geometrically, one can recognize the $SU(2)$ algebra as that corresponding to the noncommutative or fuzzy two-sphere [49, 50]. The physical size of the fuzzy two-sphere is given by

$$R = \lambda \left(\sum_{i=1}^3 \text{Tr}[(\Phi^i)^2]/N \right)^{1/2} = \pi \ell_s^2 f N \left(1 - \frac{1}{N^2} \right)^{1/2} \quad (29)$$

in the ground state solution. From the Matrix theory construction of Kabat and Taylor [51], one can infer this ground state is not simply a spherical arrangement of D0-branes rather the noncommutative solution actually represents a spherical D2-brane with N D0-branes bound to it. In the present context, the latter can be verified by seeing that this configuration has a ‘dipole’ coupling to the RR four-form. The precise form of this coupling is calculated by substituting the noncommutative scalar solution (25) into the world-volume interaction (22), which yields

$$-\frac{R^3}{3\pi g \ell_s^3} \left(1 - \frac{1}{N^2} \right)^{-1/2} \int dt F_{t123}^{(4)} . \quad (30)$$

for the ground state solution. Physically this $F^{(4)}$ -dipole moment arises because antipodal surface elements on the sphere have the opposite orientation and so form small pairs of separated membranes and anti-membranes. Of course, the spherical configuration carries no net D2-brane charge.

Given that the noncommutative ground state solution corresponds to a bound state of a spherical D2-brane and N D0-branes, one might attempt to match the above results using the dual formulation. That is, this system can be analyzed from the point of view of the (abelian) world-volume theory of a D2-brane. In this case, one would consider a spherical D2-brane carrying a flux of the $U(1)$ gauge field strength representing the N bound D0-branes, and at the same time, sitting in the background of the constant RR four-form field strength (21). In fact, one does find stable static solutions, but what is more surprising is how well the results match those calculated in the framework of the D0-branes. The results for the energy, radius and dipole coupling are the same as in eqs. (28), (29) and (30), respectively, except that the factors of $(1 - 1/N^2)$ are absent [6]. Hence for large N , the two calculations agree up to $1/N^2$ corrections.

One expects that the D2-brane calculations would be valid when $R \gg \ell_s$ while naively the D0-brane calculations would be valid when $R \ll \ell_s$. Hence it appears there is no common domain where the two pictures can both produce reliable results. However, a more careful consideration of range of validity of the D0-brane calculations only requires that $R \ll \sqrt{N} \ell_s$. This estimate is found by requiring that the scalar field commutators appearing in the full nonabelian potential (19) are small so that the

Taylor expansion of the square root converges rapidly. Hence for large N , there is a large domain of overlap where both of the dual pictures are reliable. Note the density of D0-branes on the two-sphere is $N/(4\pi R^2)$. However, even if R is macroscopic it is still bounded by $R \ll \sqrt{N}\ell_s$ and so this density must be large compared to the string scale, *i.e.*, the density is much larger than $1/\ell_s^2$. With such large densities, one can imagine the discreteness of the fuzzy sphere is essentially lost and so there is good agreement with the continuum sphere of the D2-brane picture. More discussion on the noncommutative geometry appears in Appendix A.

Finally note that the Born-Infeld action contains couplings to the Neveu-Schwarz two-form which are similar to that in eq. (22). From the expansion of $\sqrt{\det(Q)}$, one finds a cubic interaction

$$\frac{i}{3}\lambda^2 T_0 \int dt \text{Tr} (\Phi^i \Phi^j \Phi^k) H_{ijk}(t) . \quad (31)$$

Hence the noncommutative ground state, which has $\text{Tr} (\Phi^i \Phi^j \Phi^k) \neq 0$, also acts as a source of the B field with

$$-\frac{R_0^3}{3\pi g \ell_s^3} \left(1 - \frac{1}{N^2}\right) \int dt H_{123} . \quad (32)$$

This coupling is perhaps not so surprising given that the noncommutative ground state represents the bound state of a spherical D2-brane and N D0-branes. Explicit supergravity solutions describing D2-D0 bound states with a planar geometry have been found [52], and are known to carry a long-range H field with the same profile as the RR field strength $F^{(4)}$. One can also derive this coupling from the dual D2-brane formulation. Furthermore, we observe that the presence of this coupling (31) means that we would find an analogous dielectric effect if the N D0-branes were placed in a constant background H field. This mechanism plays a role in describing D-branes in the spacetime background corresponding to a WZW model [53, 54]. It seems that quantum group symmetries may be useful in understanding these noncommutative configurations [55].

The example considered above must be considered simply a toy calculation demonstrating the essential features of the dielectric effect for D-branes. A more complete calculation would require analyzing the D0-branes in a consistent supergravity background. For example, the present case could be extended to consider the asymptotic supergravity fields of a D2-brane, where the RR four-form would be slowly varying but the metric and dilaton fields would also be nontrivial. Alternatively, one can find solutions with a constant background $F^{(4)}$ in M-theory, namely the $\text{AdS}_4 \times \text{S}^7$ and $\text{AdS}_7 \times \text{S}^4$ backgrounds — see, *e.g.*, [56]. In lifting the D0-branes to M-theory, they become gravitons carrying momentum in the internal space. Hence the expanded D2-D0 system considered here correspond to the ‘giant gravitons’ of ref. [57]. The analog of the D2-D0 bound state in a constant background $F^{(4)}$ corresponds to M2-branes

with internal momentum expanding into AdS_4 [11, 58], while that in a constant H field corresponds to the M2-branes expanding on S^4 [57]. Giant gravitons will be discussed at length in the next section. Alternatively, the dielectric effect has been found to play a role in other string theory contexts, for example, in the resolution of certain singularities in the AdS/CFT correspondence [9]. Further, one can consider more sophisticated background field configurations which through the dielectric effect generate more complicated noncommutative geometries [59, 60]. There is also an interesting generalization to open dielectric branes, in which the extended brane emerging from the dielectric effect ends on another D-brane [61]. Other interesting applications of the dielectric effect for D-branes can be found in ref. [62].

5. Giant Gravitons

From the above discussion, it seems that in the M-theory backgrounds of $\text{AdS}_4 \times S^7$ or $\text{AdS}_7 \times S^4$, one will find that an M2-brane carrying internal momentum will expand into a stable spherical configuration. A Matrix theory description of such states in terms of noncommutative geometry was only developed recently [63]. Instead the original analysis of these configurations was made in terms of the abelian world-volume theory of the M2-brane. In fact, the spherical M2-branes expanding into AdS_4 were actually discovered some time ago [64]. It turns out that M5-branes will expand in a similar way for these backgrounds, and further that expanded D3-branes arise in the type IIB supergravity background $\text{AdS}_5 \times S^5$. A detailed analysis [11, 57, 58] shows that these expanded branes are BPS states with the quantum numbers of a graviton. Ref. [65] extends this discussion to more general expanded configurations. In the following, we will discuss the details of the effect for the D3-branes. Most of the discussion applies equally well for the analogous M2- and M5-brane configurations.

The line element for $\text{AdS}_5 \times S^5$ may be written as:

$$ds^2 = - \left(1 + \frac{r^2}{L^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_3^2 + L^2 \left(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\tilde{\Omega}_3^2 \right) . \quad (33)$$

This background also involves a self-dual RR five-form field strength with terms proportional to the volume forms on the two five-dimensional subspaces: $F^{(5)} = \frac{4}{L}[\varepsilon(\text{AdS}_5) + \varepsilon(S^5)]$. With the coordinates chosen above, the four-form potential on the the AdS part of the space is

$$C_{electric}^{(4)} = -\frac{r^4}{L} dt \varepsilon(S^3) \quad (34)$$

where $\varepsilon(S^3)$ is the volume form for the three-sphere described by $d\Omega_3^2$. Similarly, the

potential on the S^5 is

$$C_{\text{magnetic}}^{(4)} = L^4 \sin^4 \theta d\phi \varepsilon(\tilde{S}^3) \quad (35)$$

where $\varepsilon(\tilde{S}^3)$ is the volume form on $d\tilde{\Omega}_3^2$. For the D3-brane configurations of interest, the world-volume action in eqs. (1) and (2) reduces to:

$$S_3 = -T_3 \int d^4\sigma \sqrt{-\det(P[G])} + T_3 \int P[C^{(4)}] . \quad (36)$$

Here, the world-volume gauge field has been set to zero, which will be consistent with the full equations of motion.

Following ref. [57], one can find solutions where a D3-brane has expanded on the S^5 to a sphere of fixed θ while it orbits the S^5 in the ϕ direction. Our static gauge choice matches the spatial world-volume coordinates with the angular coordinates on $d\tilde{\Omega}_3^2$, and identifies $\sigma^0 = t$. Now we consider a trial solution of the form: $\theta = \text{constant}$, $r = 0$ and $\phi = \phi(t)$. Substituting this ansatz into the world-volume action (36) and integrating over the angular coordinates, yields the following Lagrangian

$$\mathcal{L}_3 = \frac{N}{L} \left[-\sin^3 \theta \sqrt{1 - L^2 \cos^2 \theta \dot{\phi}^2} + L \sin^4 \theta \dot{\phi} \right] . \quad (37)$$

Here we have introduced the (large positive) integer N which counts the five-form flux on S^5 . This is also, of course, the rank of the $U(N)$ gauge group in the dual super-Yang-Mills theory. Introducing the conjugate angular momentum $P_\phi = \delta \mathcal{L}_3 / \delta \dot{\phi}$, we construct the Hamiltonian:

$$\mathcal{H}_3 = P_\phi \dot{\phi} - \mathcal{L}_3 = \frac{N}{L} \sqrt{p^2 + \tan^2 \theta (p - \sin^2 \theta)^2} \quad (38)$$

where $p = P_\phi / N$. Given that the Hamiltonian is independent of ϕ , the equations of motion will be solved with constant angular momentum (and hence constant $\dot{\phi}$). For fixed p , eq. (38) can be regarded as the potential that determines the angle θ for equilibrium. Examining \mathcal{H}_3 in detail reveals degenerate minima at $\sin \theta = 0$ and $\sin^2 \theta = p$, and at either of these minima, the energy is $\mathcal{H}_3 = P_\phi / L$. The expanded configurations are then the giant gravitons of ref. [57]. An important observation is that the minima at $\sin^2 \theta = p$ only exist for $p \leq 1$. As p grows beyond $p = 1$, the minima at $\theta \neq 0$ are lifted above that at $\sin \theta = 0$ and then disappear completely if $p > 9/8$.

The discussion above indicates that one can also consider the possibility of a brane expanding into the AdS part of the spacetime [11, 58]. That is we wish to find solutions where a D3-brane has expanded to a sphere of constant r while it still orbits in the ϕ direction on the S^5 . Choosing static gauge, we again identify $\sigma^0 = t$ but match the remaining world-volume coordinates with the angular coordinates on $d\Omega_3^2$. The trial

solution is now: $\theta = 0$, $r = \text{constant}$ and $\phi = \phi(t)$. Beginning with the same[†] world-volume action (36), one calculates as before and arrives at the following Hamiltonian

$$\mathcal{H}_3 = \frac{N}{L} \left[\sqrt{\left(1 + \frac{r^2}{L^2}\right) \left(p^2 + \frac{r^6}{L^6}\right)} - \frac{r^4}{L^4} \right]. \quad (39)$$

where as before $p = P_\phi/N$. Examining $\partial\mathcal{H}_3/\partial r = 0$, one finds minima located at $r = 0$ and $(r/L)^2 = p$. The energy at each of the minima is $\mathcal{H}_3 = P_\phi/L$. In ref. [11], these expanded configurations were denoted as dual giant gravitons. An essential difference from the previous case, however, is that the minima corresponding to expanded branes persist for arbitrarily large p .

It is interesting to consider the motion of these expanded brane configurations. Evaluating $\dot{\phi}$ for any of the above solutions, remarkably one finds the same result: $\dot{\phi} = 1/L$, independent of P_ϕ . Further the center of mass motion for any of the equilibrium configurations in the full ten-dimensional background is along a null trajectory. For example, for the D3-branes expanded on S^5

$$ds^2 = -(1 - L^2 \cos^2 \theta \dot{\phi}^2) dt^2 = 0 \quad (40)$$

when evaluated for $\dot{\phi} = 1/L$ and $\theta = 0$ (= the center of mass position). This is, of course, the expected result for a massless ‘point-like’ graviton, but it applies equally well for both of the expanded brane configurations. However, note that in the expanded configurations, the motion of each element of the sphere is along a time-like trajectory.

From the point of view of five-dimensional supergravity in the AdS space, the stable brane configurations correspond to massive states with $M = P_\phi/L$. Their angular momentum means that these states are also charged under a $U(1)$ subgroup of the $SO(6)$ gauge symmetry in the reduced supergravity theory. With the appropriate normalizations, the charge is $Q = P_\phi/L$, and hence one finds that these configurations satisfy the appropriate BPS bound [57]. One can therefore anticipate that all of these configurations should be supersymmetric. The latter result has been verified by an explicit analysis of the residual supersymmetries [11, 58].

The $AdS_5 \times S^5$ background is a maximally supersymmetric solution of the type IIB supergravity equations with 32 residual supersymmetries. That is the background fields are invariant under supersymmetries parameterized by 32 independent Killing spinors. These Killing spinors are determined by setting

$$\delta\Psi_M = D_M\epsilon - \frac{i}{480} \Gamma_M^{PQRST} F_{PQRST}^{(5)} \epsilon = 0 \quad (41)$$

[†] Our conventions are such that we actually consider an anti-D3-brane here [11]. That is the sign of the Wess-Zumino term in eq. (36) was reversed.

as the variations of all of the other type IIB supergravity fields vanish automatically. The solutions take the form $\epsilon = M(x^\mu)\epsilon_0$ where ϵ_0 is an arbitrary constant complex Weyl spinor.

A supersymmetric extension of the abelian world-volume action has been constructed for D3-branes (and all other Dp-branes) in a general supergravity background [66]. This action can be viewed as a four-dimensional nonlinear sigma model with a curved superspace as the target space. Hence the theory is naturally invariant under the target-space supersymmetry. Further however, formulating the action with manifest ten-dimensional Lorentz invariance, requires an additional fermionic invariance on the world-volume called κ -symmetry. For a test brane configuration where both the target space and world-volume fermions vanish, residual supersymmetries may arise provided there are Killing spinors which satisfy a combined target-space supersymmetry and κ -symmetry transformation. The latter amounts to imposing a constraint $\Gamma\epsilon = \epsilon$ where

$$\Gamma = -\frac{i}{4!}\varepsilon^{i_1\cdots i_4}\partial_{i_1}X^{M_1}\cdots\partial_{i_4}X^{M_4}\Gamma_{M_1\cdots M_4} . \quad (42)$$

Of course, this constraint is only evaluated on the D3-brane world-volume. For all of the minima of the potentials in both eqs. (38) or (39), this constraint reduces to imposing the same projection

$$(\Gamma^{t\phi} + 1)\epsilon_0 = 0 . \quad (43)$$

Hence not only are the expanded branes and the point-like state all BPS configurations, all of these configurations preserve precisely the same supersymmetries. Note that this projection is what one might have expected for a massless particle moving along the ϕ direction, *e.g.*, one can compare this result to the supersymmetries of gravitational waves propagating in flat space [67].

The brane configurations all preserve one half of the 32 supersymmetries of the background $\text{AdS}_5 \times S^5$ spacetime. The 16 supersymmetry transformations satisfying the ‘wrong’ projection, *i.e.*, $(\Gamma^{t\phi} - 1)\epsilon_0 = 0$, would leave the background spacetime invariant but generate fermionic variations of the world-volume fields, which at the same time would leave the energy invariant. Of course, the equations of motion eliminate half of these to leave 8 fermionic zero-modes in each of the bosonic configurations studied here. These zero-modes are regarded as operators acting on a quantum space of states [68], which then build up for each bosonic configuration the full $2^8 = 256$ states of the supergraviton multiplet, as usual.

Much of the interest in giant gravitons comes from the suggestion [57] that they are related to the ‘stringy exclusion principle’ [69]. The latter arises in the AdS/CFT correspondence [4] where it is easily understood in the conformal field theory. A family of chiral primary operators in the $N=4$ super-Yang-Mills theory terminates at some

maximum weight (and R-charge) because the $U(N)$ gauge group has a finite rank. In terms of the dual AdS description, these operators are associated with single particle states and the R-charge of the operators is dual to angular momentum on the internal five-sphere. So the appearance of an upper bound on the angular momentum seems mysterious from the point of view of the supergravity theory, where there is no apparent upper bound on the Kaluza-Klein momentum. The suggestion of ref. [57] is that if the dual single particle states are identified with the giant gravitons, the D3-branes expanded on the S^5 . Then the upper bound is produced by the fact that these BPS states only exist for $p \leq 1$. In fact, this exactly reproduces the desired upper bound on the angular momentum: $P_\phi \leq N$.

Actually the correct interpretation is slightly more subtle. In [70], it was argued that the giant gravitons were dual to certain subdeterminant operators. While these can be decomposed as sums of multi-trace operators, the family of operators still terminates due to the finite rank of the $U(N)$ group with the full determinant operator. The latter carries R-charge N . Further studies [71, 72] have provided strong evidence supporting this suggestion, at least for large giant gravitons, *i.e.*, those with P_ϕ near N .

Ref. [58] provide some interesting calculations in the context of the dual CFT. They seem to be able to identify certain semi-classical field configurations with same properties as the dual giant gravitons. This suggests a picture where the D3-branes expanded on AdS_5 can be understood in terms of coherent states in the $N=4$ super-Yang-Mills theory. A complementary description in terms of large symmetric operators was suggested by [71].

6. Intersecting Branes

One interesting aspect of the (abelian) Born-Infeld action (1) is that it supports solitonic configurations describing lower-dimensional branes protruding from the original D-brane [73, 74, 75]. For example, in the case of a D3-brane, one finds spike solutions, known as ‘bions’, corresponding to fundamental strings and/or D-strings extending out of the D3-brane. In these configurations, both the world-volume gauge fields and transverse scalar fields are excited. The gauge field corresponds to that of a point charge arising from the end-point of the attached string, *i.e.*, an electric charge for a fundamental string and a magnetic monopole charge for a D-string. The scalar field describes the deformation of the D3-brane geometry caused by attaching the strings. These solutions seem to have a surprisingly wide range of validity, even near the core of the spike where the fields are no longer slowly varying. In fact, one can show that the electric spike corresponding to a fundamental string is a solution of the full string theory equations of motion [76]. Further the dynamics of these solutions, as probed through small fluctuations, agrees with the expected string behavior [77]. In part, these remarkable agreements are probably related to the fact that these are supersymmetric configurations.

For the system of N D-strings ending on a D3-brane, there is also a dual description in terms of the nonabelian world-volume theory of the N D-strings. There one finds solutions which have an interpretation, in terms of noncommutative geometry, as describing the D-strings expanding out in a funnel to become an orthogonal D3-brane. In fact, there is an extensive discussion of this system in the literature — see, *e.g.*, [78, 79] — where the emphasis was on the close connection [78] of the D-string equations to the Nahm equations for BPS monopoles [80]. In ref. [12], our emphasis was on the interpretation of these solutions in terms of noncommutative geometry and the remarkable agreement that one finds with the D3-brane spikes in the large N limit.

For N D-strings in flat space, the dynamics is determined completely by the Born-Infeld action (12) which reduces to [6, 37]

$$S = -T_1 \int d^2\sigma \text{STr} \sqrt{-\det (\eta_{ab} + \lambda^2 \partial_a \Phi^i Q_{ij}^{-1} \partial_b \Phi^j)} \det (Q^{ij}) , \quad (44)$$

where

$$Q^{ij} = \delta^{ij} + i\lambda [\Phi^i, \Phi^j] . \quad (45)$$

Implicitly here, the world-volume gauge field has been set to zero, which will be a consistent truncation for the configurations considered below. With the usual choice of static gauge, we set the world-volume coordinates: $\tau = t = x^0$ and $\sigma = x^9$. For simplicity, one might consider the leading-order (in λ) equations of motion coming from this action:

$$\partial^a \partial_a \Phi^i = [\Phi^j, [\Phi^j, \Phi^i]] . \quad (46)$$

Of course, a simple solution of these equations are constant commuting matrices, as in eq. (20). As mentioned in section (3), such a solution describes N separated parallel D-strings sitting in static equilibrium.

To find a dual description of the bion solutions of the D3-brane theory [73, 74], one needs a static solution which represents the D-strings expanding into a D3-brane. The corresponding geometry would be a long funnel where the cross-section at fixed σ has the topology of a two-sphere. In this context, the latter cross-section naturally arises as a fuzzy two-sphere [49, 50] if the scalars have values in an $N \times N$ matrix representation of the $SU(2)$ algebra (26). Hence one is lead to consider the ansatz

$$\Phi^i = \frac{R(\sigma)}{\lambda \sqrt{N^2 - 1}} \alpha^i, \quad i = 1, 2, 3, \quad (47)$$

where we will focus on case where the α^i are the irreducible $N \times N$ $SU(2)$ matrices. Then with the normalization in eq. (47), the function $|R(\sigma)|$ corresponds precisely to

the radius of the fuzzy two-sphere

$$R(\sigma)^2 = \frac{\lambda^2}{N} \sum_{i=1}^3 \text{Tr}[\Phi^i(\sigma)^2] . \quad (48)$$

Substituting the ansatz (47) into the matrix equations of motion (46) yields a single scalar equation

$$R''(\sigma) = \frac{8}{\lambda^2(N^2 - 1)} R(\sigma)^3 , \quad (49)$$

for which one simple class of solutions is

$$R(\sigma) = \pm \frac{N\pi\ell_s^2}{\sigma - \sigma_\infty} \left(1 - \frac{1}{N^2}\right)^{1/2} . \quad (50)$$

Given the above analysis, eqs. (47) and (50) only represent a solution of the leading order equations of motion (46), and so naively one expects that it should only be valid for small radius. However, one can show by direct evaluation [12] that in fact these configurations solve the full equations of motion extremizing the nonabelian action (44). The latter can also be inferred from an analysis of the world-volume supersymmetry of these configurations. Killing spinor solutions of the linearized supersymmetry conditions will exist provided that the scalars satisfy

$$D_\sigma \Phi^i = \pm \frac{i}{2} \varepsilon^{ijk} [\Phi^j, \Phi^k] . \quad (51)$$

The latter can be recognized as the Nahm equations [78]. Hence the duality between the D3-brane and D-string descriptions gives a physical realization of Nahm's transform of the moduli space of BPS magnetic monopoles. Now inserting the ansatz (47) into eq. (51) yields

$$R' = \mp \frac{2}{\lambda\sqrt{N^2 - 1}} R^2 \quad (52)$$

which one easily verifies is satisfied by the configuration given in eq. (50). Hence, one concludes that the solutions given by eqs. (47) and (50) are in fact BPS solutions preserving 1/2 of the supersymmetry of the leading order D-string theory. Now in ref. [81], it was shown that BPS solutions of the leading order theory are also BPS solutions of the full nonabelian Born-Infeld action (44).

The geometry of the solution, eqs. (47) and (50), certainly has the desired funnel shape. The fuzzy two-sphere shrinks to zero size as $\sigma \rightarrow \infty$ and opens up to fill the $x^{1,2,3}$ hypersurface at $\sigma = \sigma_\infty$. By examining the nonabelian Wess-Zumino action (13), one can show that the noncommutative solution induces a coupling to the RR four-form potential $C_{t123}^{(4)}$. This calculation confirms then that, with the minus (plus) sign in

eq. (50), the D-strings expand into a(n anti-)D3-brane which fills the $x^{1,2,3}$ directions [12]. Given that the funnel solution of the D-string theory and the bion spike of the D3-brane theory are both BPS, one might expect that there will be a good agreement between these two dual descriptions. The formula for the height of D3-brane spike above the $x^{1,2,3}$ hyperplane is [73]

$$\sigma - \sigma_\infty = \frac{N\pi\ell_s^2}{R} . \quad (53)$$

Comparing to eq. (50), one finds that for large N the two descriptions are describing the same geometry up to $1/N^2$ corrections. One finds similar quantitative agreement for large N in calculating the energy, the RR couplings and the low energy dynamics in the two dual descriptions [12]. As in the discussion of the dielectric effect, one can argue that the D3-brane description is valid for $R \gg \ell_s$ while the D-string description is reliable for $R \ll \sqrt{N}\ell_s$ [12]. Hence one can understand the good agreement between these dual approaches for large N since there is a large domain of overlap where both are reliable.

Note that in the configurations considered in this section, there are no nontrivial supergravity fields in the ambient spacetime. Hence the appearance of the noncommutative geometry in these solutions is quite distinct from that in the dielectric effect, where the external fields drive the D-branes into a certain geometry in the ground state. In the funnel solutions, the noncommutative geometry was put into the ansatz (47) by hand. An interesting extension of these solutions is then to replace the $SU(2)$ generators by those corresponding to some other noncommutative geometry, *i.e.*, to replace eq. (47) by

$$\Phi^i = \frac{R(\sigma)}{\lambda\sqrt{C}} G^i \quad (54)$$

where the G^i are new $N \times N$ constant matrices satisfying $\sum (G^i)^2 = N C$. An interesting feature of such a construction is that near the core of the funnel, the leading order equations of motion will still be those given in eq. (46). Thus for eq. (54) to provide a solution, the new generators must satisfy $[G^j, [G^j, G^i]] = 2a^2 G^i$ for some constant a , and then the radius is determined by

$$R'' = \frac{2a^2}{\lambda^2 C} R^3 , \quad (55)$$

which still has essentially the same form as eq. (49) above. Further the funnel solution of this equation also has essentially the same form as eq. (50) above, *i.e.*,

$$R = \pm \frac{\lambda\sqrt{C}}{a(\sigma - \sigma_\infty)} . \quad (56)$$

Hence the profile with $R \simeq \lambda/\sigma$ is universal for all funnels on the D-string, independent of the details of the noncommutative geometry that describes the cross-section of the funnel.

This universal behavior is curious. For example, one could consider using this framework to describe a D-string ending on an orthogonal Dp-brane with $p > 3$. However, from the dual Dp-brane formulation, one expects that for large R , solutions will essentially be harmonic functions behaving like $\sigma \propto R^{-(p-2)}$ or $R \propto \sigma^{-1/(p-2)}$. The resolution of this puzzle seems to be that the two profiles apply in distinct regimes, the first for small R and the second for large R . Hence it must be that the nonlinearity of the full Born-Infeld action will generate solutions which display a transition from one kind of behavior to another.

One particular example that we have examined in detail [13] is the case where G^i in eq. (54) are chosen to be generators describing a fuzzy four-sphere — these may be found in, *e.g.*, ref. [51]. In this case, the funnel describes the D-strings expanding into a D5-brane. One does find the expected transition in the behavior of the geometry. That is, $\sigma \approx N^{2/3}\ell_s/R$ for small R in accord with eq. (56), while at large R , higher order terms in the Born-Infeld action (44) become important yielding $\sigma \approx N^{2/3}\ell_s^4/R^3$. The same kind of behavior is also found for the corresponding solutions in the dual D5-brane world-volume theory, although of course in that case the nonlinearities of the Born-Infeld action become important for small R . An interesting feature of the D5-brane spike is that it is also nonabelian in character. Charge conservation arguments indicate that the D-string acts as a source of the second Chern class in the world-volume of the D5-brane [82]. More precisely, if N D-strings end on a collection of D5-branes, then

$$\frac{1}{8\pi^2} \int_{S^4} \text{Tr}(F \wedge F) = N , \quad (57)$$

for any four-sphere surrounding the D-string endpoint. Hence both of the dual descriptions have a noncommutative character. Again, we find that the dual constructions seem to agree at large N , however, the details of the solutions are more complex [13]. In part, the latter must be due to the fact that the D5 \perp D1 system is not supersymmetric.

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Appendix A. Noncommutative geometry

The idea that noncommutative geometry should play a role in physical theories is an old one [83]. Suggestions have been made that such noncommutative structure may resolve the ultraviolet divergences of quantum field theories, or appear in the description of spacetime geometry at the Planck scale. In the past few years, it has also become a topic of increasing interest to string theorists. From one point of view, the essential step in realizing a noncommutative geometry is replacing the spacetime coordinates by noncommuting operators: $x^\mu \rightarrow \hat{x}^\mu$. In this replacement, however, there remains a great deal of freedom in defining the nontrivial commutation relations which the operators \hat{x}^μ must satisfy. Some explicit choices that have appeared in physical problems are as follows:

(i) *Canonical commutation relations:*

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad \theta^{\mu\nu} \in \mathbb{C}$$

Such algebras have appeared in the Matrix theory description of planar D-branes [47] — for a review, see [5]. This work also stimulated an ongoing investigation by string theorists of noncommutative field theories which arise in the low energy limit of a planar D-brane with a constant B-field flux — see, *e.g.*, [45, 84, 85].

(ii) *Quantum space relations:*

$$\hat{x}^\mu \hat{x}^\nu = q^{-1} R^{\mu\nu}{}_{\rho\tau} \hat{x}^\rho \hat{x}^\tau \quad R^{\mu\nu}{}_{\rho\tau} \in \mathbb{C}$$

These algebras received some attention from physicists in the early 1990's — see, *e.g.*, [86] — and have appeared more recently in the geometry of the moduli space of $N=4$ super-Yang-Mills theory [87].

(iii) *Lie algebra relations:*

$$[\hat{x}^\mu, \hat{x}^\nu] = i f^{\mu\nu}{}_\rho \hat{x}^\rho \quad f^{\mu\nu}{}_\rho \in \mathbb{C}$$

Such algebras naturally arise in the description of fuzzy spheres as was discovered in early attempts to quantize the supermembrane [49]. These noncommutative geometries have also been applied in Matrix theory to describe spherical D-branes [51, 88]. As discussed in the main text, noncommutative geometries with a Lie-algebra structure and these

noncommutative spheres, in particular, arise very naturally in various D-brane systems since the transverse scalars are matrix-valued in the adjoint representation of $U(N)$.

Beginning with a proscribed set of commutation relations for the coordinates on a given manifold, the bulk of the problem in noncommutative geometry is to understand the algebra of functions in this framework. Of course, mathematicians are typically careful in defining the specific class of functions with which they wish to work, however, these details are usually glossed over in physical models. That is, as physicists, we are usually confident that the physics will guide the choice of functions. For a fuzzy sphere, one finds that not only is the product structure modified but that the space of functions is naturally truncated to be finite dimensional. The remainder of the discussion in the appendix will focus on these noncommutative spheres [49, 50], in part because of the emphasis they are given in the main text. We also elaborate on these examples because, in contrast to the above discussion, the natural presentation given below de-emphasizes the role of the commutation relations. In particular, the fuzzy four-sphere provides an intriguing example below.

To begin the construction of a fuzzy sphere, we begin with the standard definition of a k -sphere using the embedding in $(k+1)$ -dimensional Cartesian space

$$\sum_{i=1}^{k+1} (x^i)^2 = R^2, \quad x^i \in \mathbb{R}^{k+1}. \quad (\text{A.1})$$

Now functions on S^k can be expanded in terms of spherical harmonics as

$$f(x^i) = \sum_{\ell=0}^{\infty} f_{i_1 \dots i_\ell} x^{i_1} \dots x^{i_\ell} \quad (\text{A.2})$$

where $f_{i_1 \dots i_\ell}$ are completely symmetric and traceless tensors. We could be more precise in defining a basis of these tensors, but here we will be satisfied by noting that each term in the sum is a linear combination of spherical harmonics with principal quantum number ℓ . Showing this is straightforward: Denoting the individual terms in the sum as f_ℓ and setting aside the constraint (A.1), it is clear that the Laplacian on the Cartesian space annihilates any of these terms, *i.e.*, $\nabla^2 f_\ell = 0$. Now in spherical polar coordinates on \mathbb{R}^{k+1} , the Laplacian may be written:

$$\nabla^2 = R^{-k} \partial_R (R^k \partial_R) + R^{-2} \nabla_\Omega^2 \quad (\text{A.3})$$

where ∇_Ω^2 is the angular Laplacian on the unit k -sphere. Hence it follows that $\nabla_\Omega^2 f_\ell = \ell(\ell + k - 1) f_\ell$.

In general to produce a fuzzy sphere, one might proceed by replacing the $k+1$ continuum coordinates above by finite dimensional matrices, $x^i \rightarrow \hat{x}^i$, whose

commutation relations we leave aside for the moment. The matrices are chosen to satisfy a constraint analogous to eq. (A.1)

$$\sum_{i=1}^{k+1} (\hat{x}^i)^2 = R^2 \mathbf{1}_N . \quad (\text{A.4})$$

Similarly the continuum functions are replaced by

$$\hat{f}(\hat{x}^i) = \sum_{\ell=0}^{\ell_{max}} f_{i_1 \dots i_\ell} \hat{x}^{i_1} \dots \hat{x}^{i_\ell} \quad (\text{A.5})$$

where $f_{i_1 \dots i_\ell}$ are the same symmetric and traceless tensors consider in (A.2). Notice that the ‘noncommutative’ sum is truncated at some ℓ_{max} because for finite dimensional matrices, such products will only yield a finite number of linearly independent matrices. Thus this matrix construction truncates the full algebra of functions on the sphere to those with $\ell \leq \ell_{max}$ and the star product on the fuzzy sphere differs from that obtained by the deformation quantization of the Poisson structure on the embedding space, *i.e.*, the latter acts on the space of all square integrable functions on the sphere [89].

The simplest example of this construction is the fuzzy two-sphere, which was already encountered in sections 4 and 6. In this case, one chooses $\hat{x}^i = \lambda \alpha^i$ with $i = 1, 2, 3$ where the α^i are the generators on the *irreducible* $N \times N$ representation of $SU(2)$ satisfying the commutation relations given in eq. (26). These generators satisfy the Casimir relation

$$\sum (\alpha^i)^2 = (N^2 - 1) \mathbf{1}_N \quad (\text{A.6})$$

and so in order to satisfy the constraint (A.4), the normalization constant should be chosen as

$$\lambda = \frac{R}{\sqrt{N^2 - 1}} . \quad (\text{A.7})$$

With these $N \times N$ matrices, one finds the cutoff in eq. (A.5) is $\ell_{max} = N - 1$. So heuristically, we might say that with this construction we can only resolve distances on the noncommutative sphere for $\Delta d \gtrsim R/N$. Hence in the limit $N \rightarrow \infty$, one expects to get agreement with continuum theory, as was illustrated with the physical models in the main text.

We should mention that the entire space of functions (A.5) plays a role in the stringy constructions. To illustrate this point, we consider the example of the fuzzy two-sphere appearing in the example of the dielectric effect in section 4. In the static ground state configuration, three of the transverse scalars have a noncommutative expectation value: $\Phi^i = (f/2) \alpha^i$ for $i = 1, 2, 3$. Now one might consider excitations of this

system. In particular, it is natural to expand fluctuations of the scalars in terms of the noncommutative spherical harmonics

$$\delta\Phi^m(t) = \psi_{i_1 \dots i_\ell}^m(t) \alpha^{i_1} \dots \alpha^{i_\ell} \quad (\text{A.8})$$

where as above the coefficients $\psi_{i_1 \dots i_\ell}^m$ are completely symmetric and traceless. In the case of overall transverse scalars, *i.e.*, $m \neq 0, 1, 2, 3$, the linearized equations of motion reduce to

$$\begin{aligned} \partial_t^2 \delta\Phi^m(t) &= -[\Phi^i, [\Phi^i, \delta\Phi^m(t)]] \\ &= -\ell(\ell+1)f^2 \delta\Phi^m(t) . \end{aligned} \quad (\text{A.9})$$

Hence inserting the ansatz $\delta\Phi^m(t) \propto e^{-i\omega t}$, we find excitations with frequencies $\omega^2 = \ell(\ell+1)f^2$. Of course, these fluctuations inherit the cut off $\ell \leq \ell_{\max}$ from the noncommutative framework. The analysis of the fluctuations in the $i = 1, 2, 3$ directions is more involved but a nice description is given in [90]. In the interesting regime where the dielectric calculations are expected to be valid, *i.e.*, large N and $R \ll \sqrt{N}\ell_s$, we have $f \ll 1/\sqrt{N}\ell_s$. Hence the low energy excitations are well below the string scale, giving further corroboration that the low energy effective action provides an adequate description of the physics [6]. We might add that these frequencies match the results found from calculations in the dual continuum framework of the expanded D2-brane, again up to $1/N^2$ corrections. Of course, the latter description gives no upper cut off on the angular momentum. A similar discussion [12] applies for the excitations of the D3⊥D1-system described in section 6.

We now turn to a discussion of the fuzzy four-sphere, which is relevant for the construction of the D-string description of the D5⊥D1 system [13], mentioned at the end of section 6. This construction also played a role in ref. [51] for the Matrix theory description of spherical D4-branes (longitudinal M5-branes). At first sight, the construction of the fuzzy four-sphere appears very similar to the fuzzy two-sphere above, but in fact it yields a very different object. One begins by choosing $\hat{x}^i = \lambda G^i$ where the G^i are an appropriate set of $N \times N$ matrices. These matrices were first constructed in ref. [91] — see also ref. [51]. The G^i are given by the totally symmetric n -fold tensor product of 4×4 gamma matrices:

$$\begin{aligned} G^i &= (\Gamma^i \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma^i \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} \\ &\quad + \dots + \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \Gamma^i)_{\text{Sym}} , \end{aligned} \quad (\text{A.10})$$

where Γ_i , $i = 1, \dots, 5$ are 4×4 Euclidean gamma matrices, and $\mathbf{1}$ is the 4×4 identity matrix. The subscript ‘Sym’ means the matrices are restricted to the completely symmetric tensor product space. With the latter restriction, the dimension of the matrices becomes

$$N = \frac{(n+1)(n+2)(n+3)}{6} \quad (\text{A.11})$$

where n is the integer denoting the size of the tensor product in eq. (A.10). The ‘Casimir’ associated with the G^i matrices, *i.e.*, $G^i G^i = c \mathbf{1}_N$, is given by

$$c = n(n + 4) . \quad (\text{A.12})$$

Hence to satisfy eq. (A.4), we choose the normalization constant in $\hat{x}^i = \lambda G^i$ as $\lambda = R/\sqrt{n(n + 4)}$.

These matrices were presented as a representation of the fuzzy four-sphere on the basis of a discussion of representations of $SO(5)$ in ref. [91]. Working within Matrix theory, ref. [51] provided a series of physical arguments towards the same end. That is, the G^i produce a spherical locus, are rotationally invariant under the action of $SO(5)$ and give an appropriate spectrum of eigenvalues.

With these G^i , one can construct matrix harmonics (A.5) with $\ell \leq \ell_{max} = n$ as before [91]. However, a key difference between the fuzzy two-sphere and the fuzzy four-sphere is that the G^i do *not* form a Lie algebra (in contrast to the α^i used to construct the fuzzy two-sphere). As a result the algebra of these matrix harmonics does not close! [51, 91] In particular, one finds that the commutators $G^{ij} \equiv [G^i, G^j]/2$ define linearly independent matrices. The commutators of the G^i and G^{jk} are easily obtained[‡]:

$$\begin{aligned} [G^{ij}, G^k] &= 2(\delta^{jk} G^i - \delta^{ik} G^j), \\ [G^{ij}, G^{kl}] &= 2(\delta^{jk} G^{il} + \delta^{il} G^{jk} - \delta^{ik} G^{jl} - \delta^{jl} G^{ik}). \end{aligned}$$

Note then that the G^{ik} are the generators of $SO(5)$ rotations. Hence combined the G^i and G^{ij} give a representation of the algebra $SO(1, 5)$, as can be seen from the definition of the G^{ij} and the commutators in eq. (A.13). Hence a closed algebra of matrix functions would given by

$$\tilde{a}_{a_1 a_2 \dots a_\ell} \tilde{G}^{a_1} \tilde{G}^{a_2} \dots \tilde{G}^{a_\ell} \quad (\text{A.13})$$

where the \tilde{G}^a are generators of $SO(1, 5)$ with $a = 1, \dots, 15$, and the \tilde{a} are naturally symmetric in the $SO(1, 5)$ indices. Identifying $\tilde{G}^a = G^a$ for $a = 1 \dots 5$, the desired matrix harmonics would correspond to the subset of \tilde{a} with nonvanishing entries only for indices $a_i \leq 5$. Thus while the fuzzy four-sphere construction introduces an algebra that contains a truncated set of the spherical harmonics on S^4 , the algebra also contains a large number of elements transforming under other representations of the $SO(5)$ symmetry group that acts on the four-sphere. The reader may find a precise description of the complete algebra in ref. [91] in terms of representations of $SO(5)$ (or rather $Spin(5) = Sp(4)$).

Given this extended algebra, or alternatively the appearance of ‘spurious’ modes, one might question whether the above construction provides a suitable noncommutative description of the four-sphere. On the other hand, in the context of nonabelian D-branes

[‡] We refer the interested reader to refs. [13, 51] for more details.

(or Matrix theory), the G^i certainly form the basis for the description of physically interesting systems such as the $D5 \perp D1$ system [13]. In this case, N D-strings open up into a collection of n perpendicular D5-branes and the fuzzy four-sphere forms the cross-section of the funnel describing this geometry. As discussed after eq. (57) from the point of the D5-brane, the four-sphere is endowed with a nontrivial $SU(n)$ bundle. In this context, one can show that the relation between N and n in eq. (A.11) essentially arises from demanding that the gauge field configuration is homogeneous on the four-sphere [13]. Given that there are extra instantonic or gauge field degrees of freedom in the full physical system, it is natural that the ‘spurious’ modes above should be related to non-abelian excitations of the D5-brane theory. Evidence for this identification can be found by studying the linearized excitations of the fuzzy funnel describing the $D5 \perp D1$ system and their couplings to the bulk RR fields [13]. A thorough analysis of the noncommutative geometry [92, 93] also indicates that this identification is correct. Hence the fuzzy four-sphere construction has a natural physical interpretation within string theory.

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